W4/2 Localization, Local Properties

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Cor 2.8 Let $I \trianglelefteq A$, $S \subseteq A$ mc. set, $j: A \to S^{-1}A$ localization.
$\Rightarrow T = S + I \subseteq A/I$ is an mc. set, and

$$S^{-1}A / S^{-1}I \cong T^{-1}(A/I) \quad \text{as rings (and } A\text{-modules)}$$

Proof: $S^{-1}A / S^{-1}I \underset{\uparrow \text{C2.6(8)}}{\cong} S^{-1}(A/I) \underset{\uparrow \text{L2.7}}{\cong} T^{-1}(A/I)$, $\frac{a}{s} + S^{-1}I \longmapsto \frac{a+I}{s+I}$. □

Cor 2.9 Let $P \in \mathrm{Spec}(A)$. Then $\mathcal{Q}(A/P) \cong A_P / PA_P$, and this is called the residue field of A at P (Notation: $k(P)$).

Proof. Apply Cor 2.8 with $S = A \setminus P$. □

Exm: $\mathbb{Z}_{(p)} / p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$, $\quad \overset{\text{field}}{K[x,y]_{(x,y)} / (x,y)_{(x,y)}} \cong \mathcal{Q}(K[x,y]/(x,y)) = \mathcal{Q}(K) = K$.

$$K[x,y]_{(x)} / (x)_{(x)} \cong \mathcal{Q}\left(K[x,y]/(x)\right) \cong \mathcal{Q}(k[y]) = k(y) \ldots \text{ rational functions in } y$$

2.3 Local Properties

For $M \in A\text{-Mod}$, $P \in \mathrm{Spec}(A)$, and $j_P: M \to M_P$ the localization, $x \in M$, we often write $x_P := j_P(x) \in M_P$. Similarly for $f \in \mathrm{Hom}(M,N)$ $\rightsquigarrow f_P \in \mathrm{Hom}(M_P, N_P)$

Prop 2.10: Let $M \in A\text{-Mod}$.

(1) $\forall x \in M$: $\quad {}^{a)}\, x = 0 \overset{b)}{\Leftrightarrow} \forall P \in \mathrm{Spec}(A): x_P = 0 \overset{c)}{\Leftrightarrow} \forall P \in \mathrm{Max}(A): x_P = 0$

(2) $M = \underline{0} \Leftrightarrow \forall P \in \mathrm{Spec}(A): M_P = \underline{0} \Leftrightarrow \forall P \in \mathrm{Max}(A): M_P = \underline{0}$.

Proof: (1) a) $\Rightarrow$ b) $\Rightarrow$ c) ✓

c) $\Rightarrow$ a) Suppose $x_P = 0$ $\forall P \in \mathrm{Max}(A)$.

c) ⇒ a) Suppose $x_p = 0$ $\forall p \in \mathrm{Max}(A)$.

Let $I := \mathrm{ann}(x) := \{a \in A : ax = 0\} \trianglelefteq A$.

$$x_p = 0 \underset{\substack{\uparrow \\ x \in \mathrm{Ker}(j_p)}}{\Rightarrow} \exists s \in A \setminus p : \; sx = 0 \Rightarrow I \nsubseteq p \quad \forall p \in \mathrm{Max}(A)$$

$$\Rightarrow I = A \overset{1 \in I}{\Rightarrow} 1 \cdot x = x = 0.$$

(2) $M = 0 \Leftrightarrow \forall x \in M : x = 0 \Leftrightarrow \forall p \in \mathrm{Max}(A)\ \forall x \in M : x_p = 0$

$\Leftrightarrow \forall p \in \mathrm{Max}(A) : M_p = 0.$ □

Prop 2.11 Let $M, N \in A\text{-Mod}$, $f \in \mathrm{Hom}(M,N)$.

Then f is a mono [epi; iso]

$\Leftrightarrow \forall p \in \mathrm{Spec}(A)$: f_p is a mono [epi; iso]

$\Leftrightarrow \forall p \in \mathrm{Max}(A)$: f_p is a mono [epi; iso]

Proof: Consider the natural exact sequence

$$0 \to \mathrm{Ker}(f) \to M \xrightarrow{f} N \to \mathrm{Coker}(f) \to 0$$

$$\overset{P2.5}{\Longrightarrow} \quad 0 \to \mathrm{Ker}(f)_p \to M_p \xrightarrow{f_p} N_p \to \mathrm{Coker}(f)_p \to 0$$

is exact, so $\mathrm{Ker}(f_p) = \mathrm{Ker}(f)_p$, $\mathrm{Coker}(f)_p = \mathrm{Coker}(f_p)$.

Since f mono $\Leftrightarrow \mathrm{Ker}\, f = 0$, f epi $\Leftrightarrow \mathrm{Coker}\, f = 0$, the claim follows from Prop 2.10. □

Properties of a module (or hom) of such a form are called local properties.

2.4 Scalar Extension & Restriction

Let $f : A \to B$ be an A-algebra (= a ring hom).

- If $M \in B\text{-Mod}$, then $a \cdot m := f(a)m$ makes M into an A-module. This is restriction of scalars; functorial in M, so gives a functor $B\text{-Mod} \to A\text{-Mod}$

a functor $B\text{-}\mathrm{Mod} \to A\text{-}\mathrm{Mod}$

- As A-algebra, B is an A-module via $a \cdot b = f(a)b$.

  If $N \in A\text{-}\mathrm{Mod}$, then $B \otimes_A N$ is a B-module, via

  $b \cdot (x \otimes n) = (bx) \otimes n$.

  Gives functor $A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$, called extension of scalars.

⚠ The functors depend on f, not just on A, B. They are i.g. not inverse to each other.

Exm: •) $f: \mathbb{Z} \to \mathbb{Q}$; $\quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n_{\mathbb{Q}}$, $\quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/_{n\mathbb{Z}} = \underline{0} \quad (n \neq 0)$

•) $I \trianglelefteq A$, $\pi: A \to A/_I$, then $A/_I \otimes_A M \cong M/_{IM}$ as $A/_I$-modules.

Prop 2.12 Let $S \subseteq A$ be an mc set.

For $M \in A\text{-}\mathrm{Mod}$, $S^{-1}A \otimes_A M \cong S^{-1}M$, $\frac{a}{s} \otimes m \mapsto \frac{am}{s}$ (canonically, in $S^{-1}A\text{-}\mathrm{Mod}$)

Proof: • $S^{-1}A \times M \to S^{-1}M$, $(\frac{a}{s}, m) \mapsto \frac{am}{s}$ is A-bilinear

$\exists$ A-hom. $\varphi: S^{-1}A \otimes_A M \mapsto S^{-1}M$, $\frac{a}{s} \otimes m \mapsto \frac{am}{s}$. φ is an $S^{-1}A$-hom,

because $\varphi\left(\frac{a'}{s'} \cdot \left(\frac{a}{s} \otimes m\right)\right) = \varphi\left(\frac{a'a}{s's} \otimes m\right) = \frac{a'am}{s's} = \frac{a'}{s'} \varphi\left(\frac{a}{s} \otimes m\right)$.

φ surjective ✓ φ injective: Let $x = \sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i \in \ker(\varphi)$. Taking $\frac{a_i}{s_i} = \frac{a_i'}{s}$ w. common denominator,

$$x = \sum_{i=1}^n \frac{a_i'}{s} \otimes m_i = \sum_{i=1}^n \frac{1}{s} \otimes a_i' m_i = \frac{1}{s} \otimes \overbrace{\left(\sum_{i=1}^n a_i' m_i\right)}^{=:m} \longmapsto \frac{m}{s} = 0.$$

$\Rightarrow \exists t \in S: \; mt = 0 \quad \Rightarrow \quad x = \frac{1}{s} \otimes m = \frac{1}{st} \otimes mt = 0$.

Canonical: Let $f \in \mathrm{Hom}(M_A, N_A)$, i.e. $M \xrightarrow{f} N$. Then

$$\begin{tikzcd}
S^{-1}A \otimes_A M \arrow[r, "S^{-1}A \otimes f"] \arrow[d, "\wr"] & S^{-1}A \otimes_A N \arrow[d, "\wr"] \\
S^{-1}M \arrow[r, "S^{-1}f"] & S^{-1}N
\end{tikzcd}$$

$\frac{a}{s} \otimes m \longmapsto \frac{a}{s} \otimes f(m)$, $\quad \frac{a}{s} \otimes m \longmapsto \frac{am}{s}$, $\quad \frac{a}{s} \otimes n \longmapsto \frac{an}{s}$, $\quad m \longmapsto f(m)$

commutes.

$$S^{-1}M \longrightarrow S^{-1}N$$

$$\frac{am}{s} \qquad \frac{m}{s} \longmapsto \frac{f(m)}{s} \qquad \frac{an}{s}$$

$\square$